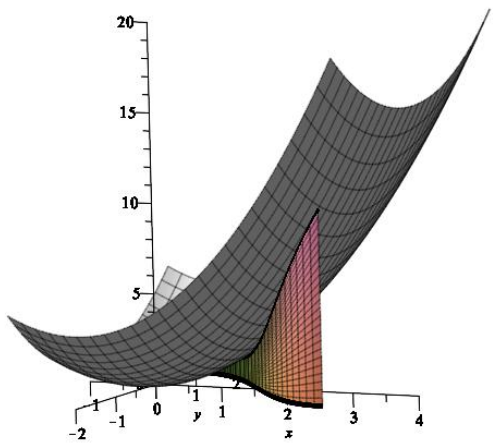
SECTION 17.2: LINE INTEGRALS

RECALL: Given a continuous function f with $f(x) \geq 0$ over an interval $[a, b]$, the area between the graph of $y = f(x)$ and the x-axis may be computed using the definite integral:

$$\text{Area} = \int_a^b f(x)\, dx$$

Suppose we have a curve C in the xy-plane beneath a surface described by a continuous function $z = f(x, y)$. Then we may interested in computing the **lateral surface area** - that is, the area between the curve C and the lift of C onto the surface as seen below.



The lateral surface area between a curve in the xy-plane, C and its lift to a surface $z = f(x, y)$ (gray).

Using the old 'chop and add' technique, we arrive at the formula below where ds is the arc length differential:

$$\text{Area} = \int_C f(x, y)\, ds$$

RECALL: Formulas for ds:

- If C is described by a smooth function $y = f(x)$: $ds = \sqrt{1 + [f'(x)]^2}\, dx$
- If C is described by a system of smooth parametric equations: $ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$
- If C is described by a smooth vector valued function $\vec{r}(t)$: $ds = \|\vec{r}\,'(t)\|\, dt = \|\vec{v}(t)\|\, dt$

GOAL: Rewrite $\displaystyle\int_C f(x, y)\, ds$ as a **single** integral in terms of just **one** variable.

NOTE: $\displaystyle\int_C 1\, ds = \int_C ds =$ the length of C

EXAMPLE 1: Let C be the portion of $y = x^2$ for $1 \le x \le 2$.

Find the lateral surface area between C and the lift of C to the surface $z = \dfrac{y}{x}$

$$\text{Ans: } \int_C \frac{y}{x}\,ds = \int_1^2 \frac{x^2}{x}\sqrt{1+(2x)^2}\,dx = \ldots = \frac{17\sqrt{17}-5\sqrt{5}}{12}\ \text{units}^2$$

EXAMPLE 2: Let C be the portion of $x^2 + y^2 = 4$ which lies in Quadrant I.

Find the lateral surface area between C and the lift of C to the surface $z = xy + 1$.

HINT: Consider parametrizing C by $x = 2\cos(t)$, $y = 2\sin(t)$, $0 \le t \le \dfrac{\pi}{2}$.

$$\text{Ans: } \int_C (xy+1)\,ds = \int_0^{\pi/2} [(2\cos(t))(2\sin(t))+1]\sqrt{(-2\sin(t))^2+(2\cos(t))^2}\,dt = \ldots = \pi + 4\ \text{units}^2$$

EXAMPLE 3: A wire is modeled by the vector valued function $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), t\rangle$, $0 \le t \le 2\pi$.

Suppose the **linear density** of the wire, in units of mass per length, is given by: $\rho(x, y, z) = \left(x^2 + y^2\right) z$.

Find and interpret $\int_C \rho(x, y, z)\, ds$

Ans: $\int_C \rho(x, y, z)\, ds = \int_0^{2\pi} \left[(2\cos(t))^2 + (2\sin(t))^2\right]\, t\, \sqrt{(-2\sin(t))^2 + (2\cos(t))^2 + (1)^2}\, dt = \ldots = 8\pi^2\sqrt{5}$

This is the mass of the wire.

RECALL: The average value of a continuous function f over an interval $[a, b]$ is given by:

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\, dx = \frac{1}{\text{length of } [a, b]} \int_{[a,b]} f(x)\, dx$$

DEFINITION: If f is continuous over C, then the average value of f over C is given by

$$\overline{f} = \frac{1}{\text{length of } C} \int_C f(x, y, z)\, ds$$

EXAMPLE 4: The temperature T at a point in the plane is given by $T(x, y) = x^2 + y^2$.

Find the average temperature on the line segment $x = 2y$, $0 \le y \le 3$.

Ans: $\overline{T} = \frac{1}{\sqrt{45}} \int_0^3 \left[(2y)^2 + y^2\right] \sqrt{1 + (2)^2}\, dy = \ldots = 15$

WORK AND CIRCULATION

Suppose $\vec{r}$ is a smooth vector-valued function which traces out the trajectory of a particle, C. Using the usual 'chop and add' approach, we find the work done moving along $\vec{r}$ through a continuous vector field $\vec{F}$ is:

$$\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot\vec{r}\,'(t)\,dt = \int_C \vec{F}\cdot d\vec{r}$$

EXAMPLE 5: Find the work done moving a particle through the field $\vec{F}(x,y,z) = \left\langle yz, x^2, xz\right\rangle$ along the line segment starting at $P(-1,0,3)$ and ending at $Q(3,-2,4)$. What happens if we start at Q and end at P?

$$\int_C \vec{F}\cdot\vec{r}\,'(t)\,dt = \int_0^1 \left\langle(-2t)(3+t),(-1+4t)^2,(-1+4t)(3+t)\right\rangle\cdot\langle 4,-2,1\rangle\ dt = ... = -\frac{31}{2}$$

Reversing direction gives $\dfrac{31}{2}$.

NOTE: If $-C$ denotes the curve C with opposite orientation, then $\displaystyle\int_{-C}\vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$.

CIRCULATION If C is a simple closed curve and $\vec{F}$ is a velocity flow field, the **circulation** of $\vec{F}$ along C is:

$$\oint_C \vec{F}\cdot\hat{T}\,ds = \oint_C \vec{F}\cdot\vec{r}\,'(t)\,dt = \oint_C \vec{F}\cdot d\vec{r}$$

EXAMPLE 6: Find the circulation of the field $\vec{F}(x,y) = \langle 2y, -x\rangle$ once around the Unit Circle, counter-clockwise.

Ans: $\displaystyle\oint_C \vec{F}\cdot\vec{r}\,'(t)\,dt = \int_0^{2\pi}\langle 2\sin(t), -\cos(t)\rangle\cdot\langle-\sin(t),\cos(t)\rangle\ dt = ... = -3\pi$

2-D FLUX

While the concepts of **work** and **circulation** are concepts that calculate accumulation of a field **along** a curve, the concept of **flux** calculates an accumulation of a field **across** a curve. For this idea to make the sense, we are restricted to curves in the plane. Later on, we'll generalize flux to three dimensions using surfaces.

RECALL: If $\vec{r}(t) = \langle x(t), y(t)\rangle$ is a smooth vector-valued function, then $\hat{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \dfrac{\langle x'(t), y'(t)\rangle}{\sqrt{[x'(t)]^2 + [y'(t)]^2}}$.

We have **two** choices for normal vectors $\hat{n}(t)$:

$$\hat{n}(t) = \frac{\langle -y'(t), x'(t)\rangle}{\sqrt{[x'(t)]^2 + [y'(t)]^2}}, \quad \text{or} \quad \hat{n}(t) = \frac{\langle y'(t), -x'(t)\rangle}{\sqrt{[x'(t)]^2 + [y'(t)]^2}}$$

If C is a simple closed curve oriented **counter-clockwise**, it can be shown that:

- $\hat{n}_{\text{in}}(t) = \dfrac{\langle -y'(t), x'(t)\rangle}{\sqrt{[x'(t)]^2 + [y'(t)]^2}}$ always points **into** the region bounded by C

- $\hat{n}_{\text{out}}(t) = \dfrac{\langle y'(t), -x'(t)\rangle}{\sqrt{[x'(t)]^2 + [y'(t)]^2}}$ always points **outward** from the region bounded by C

NOTE: Draw some pictures to convince you of the above claim.

It is customary to default to use $\hat{n}(t) = \hat{n}_{\text{out}}(t)$ if not otherwise specified.

FLUX If C is a simple closed curve and $\vec{F}$ is a velocity flow field, the **flux** of $\vec{F}$ across C is:

$$\int_C \vec{F}\cdot\hat{n}\,ds = \int_C \vec{F}\cdot\big\langle y'(t), -x'(t)\big\rangle\ dt = \int_C \vec{F}\cdot d\vec{n}$$

EXAMPLE 7: Let $\vec{F}(x, y) = \langle 2y, xy\rangle$ and let C be the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, 2)$, oriented counter-clockwise. Find the outward flux of $\vec{F}$ across C.

Ans: $\displaystyle\int_0^1 \langle 0, 0\rangle \cdot < 0, -1 >\ dt + \int_0^1 \langle 2(2t), 2t(4-4t)\rangle \cdot < 2, 4 >\ dt + \int_0^1 \langle 2(2-2t), 0\rangle \cdot < -2, 0 >\ dt = ... = \frac{16}{3}$

DIFFERENTIAL FORMS OF LINE INTEGRALS

Suppose $\vec{F}(x,y) = \langle M(x,y).N(x,y)\rangle$ and $\vec{r}(t) = \langle x(t), y(t)\rangle$. Then we may write:

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_C \vec{F}\cdot\vec{r}\,'(t)dt \\
&= \int_C \langle M(x,y), N(x,y)\rangle \cdot \left\langle x'(t), y'(t)\right\rangle dt \\
&= \int_C M(x,y)\,x'(t)\,dt + N(x,y)\,y'(t)\,dt \\
\int_C \vec{F}\cdot d\vec{r} &= \int_C M\,dx + N\,dy
\end{aligned}$$

In three dimensions, if $\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$, then $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy + P\,dz$

EXAMPLE 8: Find $\displaystyle\int_C x\,dy - y\,dx$ where C is the portion of the parabola $y = x^2$, $0 \le x \le 2$.

Ans: $\displaystyle\int_C x\,dy - y\,dx = \int_0^2 x(2x\,dx) - x^2\,dx = \ldots = \frac{8}{3}$

EXAMPLE 9: Find $\displaystyle\int_C 3z\,dx - 2x\,dy + y\,dz$ where C is the helix: $\vec{r}(t) = \langle 3\cos(t), 2\sin(t), 4t\rangle$, $0 \le t \le \pi$.

Ans: $\displaystyle\int_C 3z\,dx - 2x\,dy + y\,dz = \int_0^{\pi} 3(4t)(-3\sin(t)\,dt) - 2(3\cos(t))(2\cos(t)\,dt) + 2\sin(t)(4\,dt) = -42\pi + 16$

2 -D WORK / CIRCULATION INTEGRALS IN DIFFERENTIAL FORM: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy$

2-D (OUTWARD) FLUX INTEGRALS IN DIFFERENTIAL FORM: $\displaystyle\int_C \vec{F}\cdot d\vec{n} = \int_C M\,dy - N\,dx$

HOMEWORK: Section 17.2: 17 - 65 every odd, 73.